constituted at the point $A$, a common perpendicular in two distinct points, which is less than any assignable length $R$.

And hence follows thirdly, that (in this hypothesis) there must be a certain determinate acute angle $BAX$, drawn under which $AX$ so approaches ever more to $BX$, that only at an infinite distance does it meet it.

But further that this $AX$ is a limit in part from within in part from without of each of the aforesaid classes of straights is proved thus. First, it agrees with those straights which meet $BX$ at a finite distance since it also finally meets ; but it differs, because it meets only at an infinite distance.

But secondly it also agrees, and at the same time differs from those straights which have a common perpendicular in two distinct points with $BX$ ; because it also has a common perpendicular with $BX$ ; but in one and the same point $X$ infinitely distant. But this latter ought to be considered demonstrated in Proposition XXVIII., as I point out in its corollary.

Therefore it holds, that (in the hypothesis of acute angle) there will be a certain determinate acute angle $BAX$, drawn under which $AX$ only at an infinite distance meets $BX$, and thus is a limit in part from within, in part from without ; on the one hand of all those which under lesser acute angles meet the aforesaid $BX$ at a finite distance ; on the other hand also of the others which under greater acute angles, even to a right angle inclusive, have a common perpendicular in two distinct points with $BX$.                    Quod erat etc.

|To be Continued.|

# ON THE BEST METHOD OF SOLVING THE MARKINGS OF JUDGES OF CONTESTS.

By F. R. MOULTON.

The fact that many different methods are in use for deciding from the markings of judges the relative standing of the participants in oratorical and similar contests, and that several different methods have been stated to be the best by Professors of Mathematics in some of our colleges, may be taken as the excuse for this paper. It is questionable whether such a problem can be solved by perfectly rigorous mathematical processes, but it has seemed that a method similar to that of Hagen in the theory of probability may be applied with advantage. Let us agree to adopt the following hypotheses.

(1) The judges mark each contestant independently and by the same scale.

(2) There is a true marking for each contestant as compared to a fixed ideal.

(3) The deviation of each judge's markings from these true markings, are the result of a very great many influences, such as, the inherited inclinations re-

sulting from the entire experience of the judge's ancestry, the judge's own experiences, his business or profession, health, etc., etc.

These influences will have different degrees of importance. Let us select a unit in terms of which all the influences acting upon every judge can be expressed. Then we shall consider that an influence whose importance is $k$ times the unit influence is equivalent to $k$ unit influences working together. With this understanding the whole number of influences is some large number, $\mu$. The coefficients of those which do not operate in the case of particular judges are zero.

(4) In the long run the unit influence will be just as liable to cause the judge to err in one direction as the other. (This is plainly a pure assumption but is the only one that can be made.)

Let us now consider the question in two parts ; first, an investigation of the law of distribution of errors, and second, the best way of eliminating their influence.

I. *Investigation of the expression for the probability of the existence of an error of a given magnitude.*

By hypothesis (4) the probability that a unit influence will give a positive error is $p=\frac{1}{2}$, and negative $q=\frac{1}{2}$.

Then all possible combinations of positive and negative errors is given by the terms of the expansion

$$(1) \qquad (p+q)^{\mu}=1=\Sigma \frac{\mu !}{m ! \, n !}p^m q^n$$

where $m+n=\mu$. The general term is

$$(2) \qquad T_n=\frac{\mu !}{m ! \, n !}p^m q^n$$

which is the probability of $m$ positive and $n$ negative errors. By Stirling's theorem

$$\mu ! = \mu^{\mu} e^{-\mu} \sqrt{2\pi\mu}.$$

Therefore (2) becomes

$$(3) \qquad T_n=\left(\frac{\mu p}{m}\right)^m \left(\frac{\mu q}{n}\right)^n \frac{\sqrt{\mu}}{\sqrt{(2\pi\mu)}}=\left(\frac{\mu p}{m}\right)^{m+\frac{1}{2}}\left(\frac{\mu q}{n}\right)^{n+\frac{1}{2}} \frac{1}{\sqrt{(2\pi\mu pq)}}.$$

We now find what values of $m$ and $n$ make $T_n$ a maximum. Changing $n$ to $n-1$ and $n+1$ successively in (2) we have

$$(4) \qquad \begin{cases} T_{n-1} = \dfrac{\mu !}{(m+1)! \, (n-1)!} p^{m+1} q^{n-1} = T_n \cdot \dfrac{n}{m+1} \dfrac{p}{q} \\[2ex] T_{n+1} = \dfrac{\mu !}{(m-1)! \, (n+1)!} p^{m-1} q^{n+1} = T_n \cdot \dfrac{m}{n+1} \dfrac{q}{p} \end{cases}$$

If $T_n$ is a maximum $T_n > T_{n\pm 1}$. Therefore

(5)
$$\begin{cases} 1 > \dfrac{n}{m+1}\dfrac{p}{q} \text{ and } 1 > \dfrac{m}{n+1}\dfrac{q}{p} \\ m+n=\mu \text{ and } q=1-p. \end{cases}$$

Eliminating $n$ and $q$ we have

$$1 > \frac{\mu-m}{m+1}\frac{p}{1-p} \text{ and } 1 > \frac{m}{\mu-m+1}\frac{1-p}{p}.$$

This may be written

$$m/\mu > p + \frac{m}{\mu(\mu+1)} - \frac{1}{\mu+1} \text{ and } m/\mu < p + \frac{m}{\mu(\mu+1)}.$$

Hence as $\mu$ increases we have more and more nearly $m=\mu p$ and similarly $n=\mu q$ or $(\mu p/m)=1$ and $(\mu q/n)=1$.

Substituting these values in (3) we have

$$T_n(max)=G=\frac{1}{\sqrt{2\pi\mu pq}} \text{ and}$$

(6)
$$T_n = G\left(\frac{\mu p}{m}\right)^{m+\frac{1}{2}}\left(\frac{\mu q}{n}\right)^{n+\frac{1}{2}}$$

Let $G_{n\pm l}$ be the value which $T_{n\pm l}$ takes for $m=\mu p$; $n=\mu q$. Therefore,

(7)
$$G_{n-l}=G[1+(l/m)]^{-m-l-\frac{1}{2}}[1-(l/n)]^{-n+l-\frac{1}{2}}.$$

Since we always have $|l| < m$ we have

$$[1+(l/m)]^{-m-l-\frac{1}{2}}=e^{(-m-l-\frac{1}{2})\log(1+l/m)}=e^{[(-m-l-\frac{1}{2})/m]+[(ml^2+l^3+\frac{1}{3}l^2)/(2m^2)]\ldots\ldots}$$

Substituting in (7) we obtain

(8)
$$\begin{cases} G_{n-l}= G.e^{\left\{\frac{-l^2+l}{2m}+\frac{l^3+\frac{3}{2}l^2}{m^2}\cdots\cdots\frac{-l^2-l}{2n}+\frac{-l-\frac{3}{2}l^2}{6n^2}\cdots\cdots\right\}} \\ G_{n+l}= G.e^{\left\{\frac{-l^2-l}{2m}+\frac{-l^3-\frac{3}{2}l^2}{6m^2}\cdots\cdots\frac{-l^2-l}{2n}+\frac{l^3+\frac{3}{2}l^2}{6n^2}\cdots\cdots\right\}} \end{cases}$$

The number of positive errors in $G_{n-l}$ is

(9)
$$\begin{cases} \mu p + l \text{ and negative } \mu q - l. \quad \text{In } G_{n+l} \text{ the positive are} \\ \mu p - l \text{ and negative } \mu q + l. \end{cases}$$

By our hypothesis (4) $p=q=\frac{1}{2}$. Then we get from (8), neglecting terms higher than the first order in $1/\mu$,

(10) $$G_{n-l}=G_{n+l}=Ge^{-l^2/\frac{1}{2}\mu}.$$

Suppose that the absolute value of the error produced by a unit influence is $\alpha$, then the errors in this case are by (9),

(11)
$$\begin{cases} (\frac{1}{2}\mu+l)\alpha-(\frac{1}{2}\mu-l)\alpha=2l\alpha=x \\ (\frac{1}{2}\mu-l)\alpha-(\frac{1}{2}\mu+l)\alpha=-2l\alpha=-x. \end{cases}$$

Increasing $l$ by unity increases the right side by $2\alpha=\triangle x$, a very small quantity from the nature of the problem. From $2\alpha=\triangle x$ and (11) we have

(12) $$l=x/\triangle x.$$

Hence (10) becomes

(13) $$G_{n-l}=G_{n+l}=Ge^{-\{x^2/[(\frac{1}{2}\mu)\triangle x^2]\}}$$

We may, without loss of generality, suppose $\alpha$ and consequently $x$ to become indefinitely small, and at the same time $\mu$ will become indefinitely great. The absolute value of one unit error is $\frac{1}{2}\triangle x$, and if they are all of the same character their sum, $\mu(\triangle x/2)$, which is the maximum error possible, becomes from the nature of the problem an indefinitely large quantity. Hence $(\frac{1}{2}\mu)(\triangle x)^2$ equals a finite constant, say, $1/h^2$. Then we have

(14) $$G=(h/\sqrt{\pi})dx$$

when $\triangle x$ becomes $dx$. This is the probability of the error zero. Substituting in (13) we have

(15) $$G_{n-l}=G_{n+l}=(h/\sqrt{\pi})e^{-h^2x^2}dx=px,$$

which is the probability of an error with the magnitude $x$. This is the well-known probability formula.

II. *The method of obtaining the most probable standing of each contestant.*

Let $w_1, w_2 \ldots w_n$ be the markings given to a contestant by the respective judges. Let $z$ be the true marking which the contestant deserves.

Let $z-w_1=\varepsilon_1$; $z-w_2=\varepsilon_2 \ldots z-w_n=\varepsilon_n$. Then by (15) the probability that the error $\varepsilon_i$ will occur is

$$p\varepsilon_i = \frac{h}{\sqrt{\pi}} e^{-h^2\varepsilon_i^2} d\varepsilon_i.$$

The probability that the errors $\varepsilon_1, \varepsilon_2 \ldots \varepsilon_n$ will occur together in one marking of each judge is,

(16) $$P=p\varepsilon_1.p\varepsilon_2 \ldots p\varepsilon_n = \frac{h^n}{(\sqrt{\pi})^n} e^{-h^2(\varepsilon_1^2+\varepsilon_2^2+\ldots+\varepsilon_n^2)} d\varepsilon_1 d\varepsilon_2 \ldots d\varepsilon_n$$

The most probable value of $z$ is that one which will make $P$ a maximum. $P$ is a maximum when $(\varepsilon_1{}^2 + \varepsilon_2{}^2 + \ldots \ldots \varepsilon_n{}^2)$ is a minimum ; or, by equating to zero, the first derivative, when $\varepsilon_1 + \varepsilon_2 + \ldots \ldots \varepsilon_n = 0$. That is when $z - w_1 + z - w_2 + \ldots \ldots z - w_n = 0$. Solving for $z$ we find

$$(17) \qquad\qquad z = (w_1 + w_2 + \ldots \ldots w_n)/n.$$

Hence it appears that in the long run the best way to solve the markings of judges of contests is simply to take the arithmetical means of the markings when the merits of the contestants are compared with ideal standards.

The methods for successive approximations have already been discussed in this journal.

*The University of Chicago.*

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# A METHOD OF DEFINING THE ELLIPSE, HYPERBOLA AND PARABOLA AS CONIC SECTIONS.

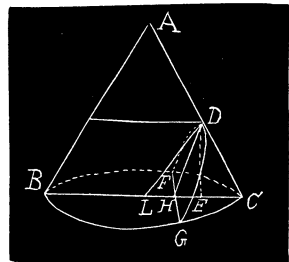By W. W. LANDIS, A. M., Professor of Mathematics in Dickinson College, Carlisle, Pa.

$ABC$ is a right circular cone, the angle at the vertex being $2\alpha$. $DFG$ is a plane section making an angle $\theta$ with the axis of the cone. Take $D$ as the origin, $DH$ as the axis of $x$, and a perpendicular through $D$ as the axis of $y$. We seek to find a relation between $x$ and $y$, the parameters being $g(=AD)$ and $\alpha$, and the variable parameter $\theta$. In the circle $BGC$, $y^2 = (a \pm b)c = ac \pm bc \ldots \ldots (1)$, where $a = BL$, $b = LH$, and $c = HC$. In the isosceles triangle $DLC$, $x^2 = d^2 \mp bc$ $\ldots \ldots (2)$, where $d = DC = DL$. Adding (1) and (2), $x^2 + y^2 = ac + d^2 \ldots \ldots (3)$.



Now $c = x\sin\theta + d\sin\alpha$,
$d = x\cos\theta\sec\alpha = x\cos\theta/\cos\alpha$, and
$a = 2g\sin\alpha$.

Making this substitution we get

$$x^2 + y^2 = x^2\frac{\cos^2\theta}{\cos^2\alpha} + 2g\sin\alpha\left[x\sin\theta + x\frac{\cos\theta\sin\alpha}{\cos\alpha}\right]$$

$$\text{or } x^2\left[1 - \frac{\cos^2\theta}{\cos^2\alpha}\right] + y^2 - 2gx\sin\alpha[\sin\theta + \cos\theta\tan\alpha] = 0 \ldots \ldots (4)$$

which we may write

$$x^2\left[\frac{\sin^2\theta - \sin^2\alpha}{1 - \sin^2\alpha}\right] + y^2 - 2gx\sin\alpha[\sin\theta + \cos\theta\tan\alpha] = 0 \ldots \ldots (5).$$